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HOMEOMORPHISMS OF S^3 LEAVING A HEEGAARD SURFACE INVARIANT

BY

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ABSTRACT. We find a finite set of generators for the group $\mathcal{H}_g$ of isotopy classes of orientation-preserving homeomorphisms of the 3-sphere S^3 which leave a Heegaard surface T of genus g in S^3 invariant. We also show that every element of the group $\mathcal{H}_g$ can be represented by a deformation of the surface T in S^3 of a very special type: during the deformation the surface T is the boundary of the regular neighborhood of a graph embedded in a fixed 2-sphere. The only exception occurs when a subset of the graph contained in a disc on the 2-sphere is "flipped over."

INTRODUCTION

A closed orientable surface T of genus g embedded in the 3-sphere S^3 is a Heegaard surface if it divides S^3 into homeomorphic handlebodies. The object of our investigation in this paper is the group $\mathcal{H}_g$ of isotopy classes of orientation-preserving homeomorphisms of S^3 which leave the surface T and each handlebody invariant. Our main result shall be to find a finite set of generators for $\mathcal{H}_g$.

The group $\mathcal{H}_g$ is of interest to topologists because it plays a central role in the equivalence of Heegaard sewings of a 3-manifold. This was suggested by work of R. Craggs **[C]** and in a joint paper by J. Birman and the author **[B-P]**. The understanding of this group may contribute to the classification problem of 3-manifolds via Heegaard sewings. Moreover, the elements of the group $\mathcal{H}_g$ determine deformations of handlebodies in the 3-sphere bringing the handlebodies back to their initial position. The group of such deformations is an extension of $\mathcal{H}_g$ by $\mathbf{Z}_2$.

Previous results on the group $\mathcal{H}_g$ concern the classical cases of $\mathcal{H}_0$ which is the trivial group and $\mathcal{H}_1$ which is the group $\mathbf{Z}_2$. The group $\mathcal{H}_g$ is infinite for $g \geqslant 2$ and Goeritz **[G]** found, in 1935, a finite set of generators for $\mathcal{H}_2$. To our knowledge the group $\mathcal{H}_g$ has not been studied for $g \geqslant 3$.

After describing in Chapter 1 some different definitions of the group $\mathcal{H}_g$,

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we define in Chapter 2 five elements in $\mathcal{H}_g$ (four of these generate a subgroup $\mathcal{P}_g$ which will play an important role in our argument). In Chapter 3 we find a finite set of generators for another important subgroup $\mathcal{H}_g^i$, the group of "slidings of the ith handle". Using these two subgroups we are able to show that the elements defined in Chapter 2 generate the group $\mathcal{H}_g$ (see Theorem 1 and Corollary 1 in Chapter 4).

The heart of the argument occurs in Lemma 3 of Chapter 3, and in Chapter 4. In Chapter 5 we obtain a second set of generators of the group $\mathcal{H}_g$ (see Corollary 2). We also get an interesting interpretation of elements of the group $\mathcal{H}_g$. Any element of $\mathcal{H}_g$ is the isotopy class of an orientation-preserving homeomorphism of S^3 which leaves the surface T invariant. This homeomorphism is isotopic to the identity as a homeomorphism of S^3. During the isotopy the surface may pass through a variety of positions in which it appears to be "knotted.". By Corollary 3 of Chapter 5 such "apparently knotted" embeddings can be avoided in the sense that the isotopy can be chosen to be a sequence of special isotopies which are particularly simple. In Chapter 6 we discuss briefly the homomorphism from the group $\mathcal{H}_g$ to the outer automorphism group of the fundamental group of one of the handlebodies bounded by the surface T.

I am most indebted to my thesis adviser, Joan Birman, for her constant help and insights.

Chapter 1. Different Ways of Defining $\mathcal{H}_g$

In this chapter we define the group $\mathcal{H}_g$ and give several different interpretations of its meaning. In order to do that, we first establish some notation and definitions. We assume throughout $g > 1$.

A *special neighborhood* $N(G)$ is a regular neighborhood of a graph G embedded in S^3, such that $N(G)$ is a union of *beams*, i.e. embeddings of $D^2 \times I$, and *nodes*, i.e. embeddings of the 3-ball D^3, with the following restrictions: (i) the interiors of the beams and the nodes are disjoint, (ii) the discs $D^2 \times \partial I$ in the beams are identified with disjoint discs on the boundaries of the nodes, (iii) the cores of the beams are subsets of the edges of G, (iv) the centers of the nodes are vertices of G. If a solid handlebody is a special neighborhood, a *meridian disc* of the handlebody is the embedding of the disc $D^2 \times \frac{1}{2}$ in a beam.

To describe embeddings in the 3-sphere, we will view S^3 as the union of $\mathbf{R}^3$ and a point at infinity. For any natural number g, the *standard graph* G is a bouquet of g loops in the y-z plane, where the ith loop is symmetric with respect to a line whose polar coordinate is $\theta = 2\pi(i-1)/g$. The graph G, for $g = 4$, is illustrated in Figure 1. The *standard handlebody* N of genus g is a special neighborhood of G. Finally, the *standard surface* T of genus g is the boundary of N.

The standard graph has a single vertex, hence N has a unique node which we denote by the symbol $\overline{N}$. The ith *handle* H_i is the beam through which the edge e_i passes. The meridian disc of the ith handle will be denoted s_i, and the

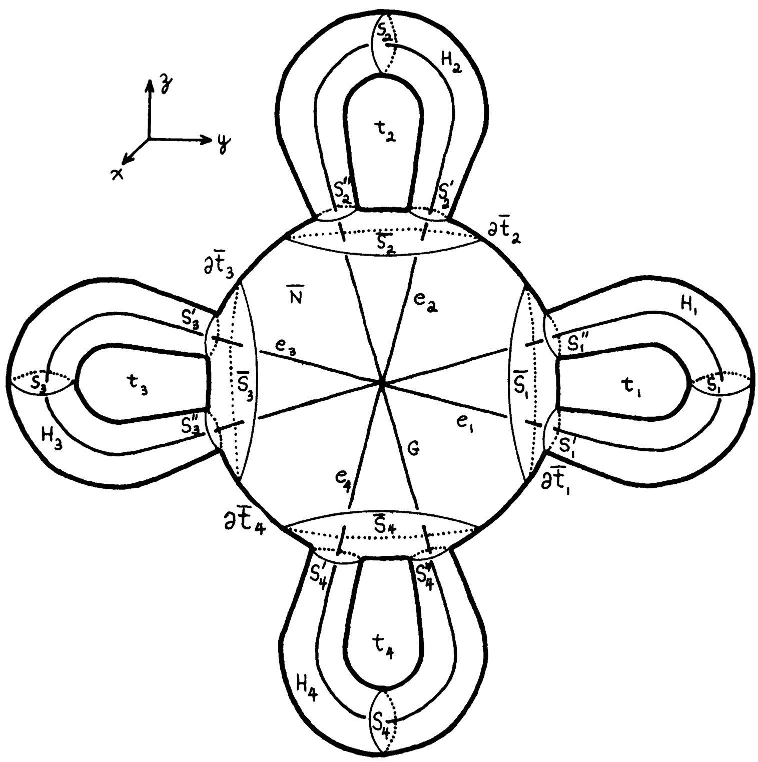

FIGURE 1. The standard handlebody of genus 4.

two discs in the intersection of H_i and $\bar{N}$ will be called the *feet* of the ith handle. The *first foot* s_i' and the *second foot* s_i'' are ordered according to increasing polar coordinates. The disc t_i will be the component of the intersection of the complement of N and the y-z plane which intersects the meridian disc s_i. The ith handle is contained in a 3-ball B_i in such a way that $B_i \cap T$ divides the boundary of B_i into two discs, $\bar{s}_i$ and $\bar{t}_i$; the disc $\bar{s}_i$ is in N, and is called the *pedestal* of the ith handle, and $\bar{t}_i$ is in the complement of N.

We may now define our group $\mathcal{H}_g$ and relate $\mathcal{H}_g$ to other groups. We denote the group of orientation-preserving homeomorphisms of the topological space X by $H(X)$ and the ith homotopy group of X by $\pi_i(X)$. The group $\mathcal{H}_g$ is defined as $\pi_0(H(S^3, N))$. A homeomorphism in the group $H(S^3, N)$ leaves N and the closure of the complement of N, which we denote $\mathrm{Cl}(S^3 - N)$, invariant, therefore

$$\mathcal{H}_g \approx \pi_0(H(S^3, N)) \approx \pi_0(H(S^3, \mathrm{Cl}(S^3 - N))).$$

The group $\mathcal{H}_g$ arises naturally as a subgroup of the mapping class groups $\pi_0(H(N))$ and $\pi_0(H(\mathrm{Cl}(S^3 - N)))$ of the handlebodies N and $\mathrm{Cl}(S^3 - N)$, and also as a subgroup of the mapping class group $\pi_0(H(T))$ of the surface T (see **[M]**). The inclusions of $\mathcal{H}_g$ into these three groups are induced by the restrictions of the homeomorphisms in $H(S^3, N)$ to homeomorphisms in

$H(N)$, $H(\mathrm{Cl}(S^3 - N))$ and $H(T)$, respectively, followed by a quotient map to the corresponding group of isotopy classes. Clearly, as a subgroup of the mapping class group of the surface T, the group $\mathcal{H}_g$ is exactly the intersection of the subgroup induced by the inclusions of $\pi_0(H(N))$ and of $\pi_0(H(\mathrm{Cl}(S^3, N)))$ in the mapping class group of the surface.

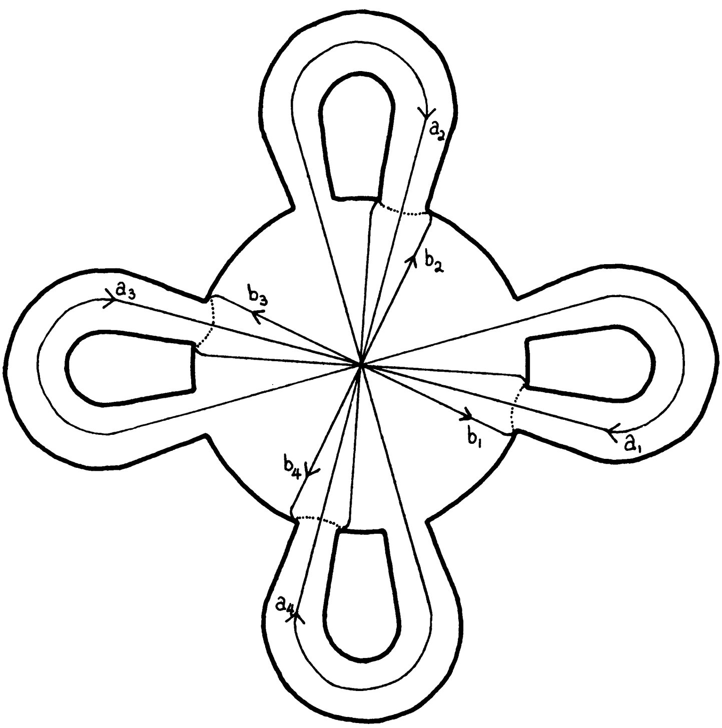

FIGURE 2. The generators of $\pi_1(T)$.

Our group $\mathcal{H}_g$ may also be defined algebraically. The fundamental group $\pi_1(T)$ of the surface T is generated by the elements $a_1, \ldots, a_g, b_1, \ldots, b_g$ represented by the loops in Figure 2, and admits the presentation

$$\left\langle a_1, \ldots, a_g, b_1, \ldots, b_g \mid \left[a_1 b_1 a_1^{-1} b_1^{-1}\right] \cdots \left[a_g b_g a_g^{-1} b_g^{-1}\right]\right\rangle.$$

By a fundamental result due to J. Neilsen **[B-1**, p. 148] the mapping class group of the surface T is isomorphic to the outer automorphism group of $\pi_1(T)$. If we denote the normal subgroups generated by $\{a_1, \ldots, a_g\}$ and $\{b_1, \ldots, b_g\}$ in $\pi_1(T)$ by $N(a)$ and $N(b)$, respectively, then McMillan **[M]** has shown that the group $\pi_0(H(N))$ has an interpretation as $\{\varphi_* \in \mathrm{Out}^+(\pi_1(T)) \mid \varphi_*(N(b)) = N(b)\}$ and the group $\pi_0(H(\mathrm{Cl}(S^3 - N)))$ has an interpretation as $\{\varphi_* \in \mathrm{Out}^+(\pi_1(T)) \mid \varphi_*(N(a)) = N(a)\}$. Since our group $\mathcal{H}_g$ may be considered as the intersection of these two groups, we have

$$\mathcal{H}_g = \left\{\varphi_* \in \mathrm{Out}^+(\pi_1(T)) \mid \varphi_*(N(a)) = N(a) \quad \text{and} \quad \varphi_*(N(b)) = N(b)\right\}.$$

A very useful and interesting interpretation of the group $\mathcal{H}_g$ is by way of equivalence classes of deformations of S^3. All homeomorphisms of S^3 are isotopic, i.e. $\pi_0(H(S^3)) = 0$, therefore, instead of using a homeomorphism in $H(S^3, T)$ to represent an element in $\mathcal{H}_g$, we may use as representative a deformation D: $I \to H(S^3)$, with the properties that $D(0)$ is the identity homeomorphism and $D(1)$ is a homeomorphism in $H(S^3, T)$. We now formalize this point of view. Let two homeomorphisms φ and φ' of S^3 be equivalent if and only if $\varphi(N) = \varphi'(N)$, and let $K(N)$ be the quotient space of $H(S^3)$ generated by this equivalence relation with the quotient space topology. We then have the following sequence of continuous maps:

$$H(S^3, N) \xrightarrow{i} H(S^3) \xrightarrow{\pi} K(N)$$

where the map i is the natural injection and the map π the projection defined by the equivalence relation. The map π is a fibration, therefore a long exact sequence of homotopy groups is associated to this fibration:

$$\to \pi_1(H(S^3, N)) \to \pi_1(H(S^3)) \to \pi_1(K(N)) \to \pi_0(H(S^3, N)) \to \pi_0(H(S^3)).$$

In order to understand the sequence better we review known facts: some of the groups in the sequence are known, $\pi_0(H(S^3)) = 0$, $\pi_1(H(S^3)) = \mathbf{Z}_2$ and, by definition, $\pi_0(H(S^3, N)) = \mathcal{H}_g$. We also note that

$$\pi_1(H(S^3, N)) \xrightarrow{i_*} \pi_1(H(S^3))$$

must be the zero map: the nonzero element $\alpha \in \pi_1(H(S^3))$ is represented by a loop of deformations of S^3 that leaves no point fixed because $\pi_1(H(\mathbf{R}^3)) = 0$, but any loop representing an element of the image is isotopic to one which leaves N pointwise invariant because $\pi_1(H(N)) = 0$. Using these facts we get a short exact sequence

$$0 \to \mathbf{Z}_2 \to \pi_1(K(N)) \to \mathcal{H}_g \to 0.$$

The group $\mathcal{H}_g$ can now be described as a quotient by $\mathbf{Z}_2$ of $\pi_1(K(N))$. An element in $\pi_1(K(N))$ is represented by a loop in $K(N)$ which can be lifted to an arc in $H(S^3)$. Such an arc will be denoted a *special deformation* and may be defined as a deformation D: $I \to H(S^3)$ such that $D(t) = \varphi_t$, $t \in I$, with the properties $\varphi_0 = \mathrm{id}_{S^3}$ and $\varphi_1(N) = N$ setwise.

Therefore, the elements in the group $\mathcal{H}_g$ may be represented by continuous sequences of embeddings of a handlebody in S^3. If D is a special deformation then $\varphi_t(N)$ is an embedding of a handlebody in S^3, such that the complement is also a handlebody since the deformation starts off with $\varphi_0(N) = N$. Therefore, all the embeddings $\varphi_t(N)$ determine Heegaard splittings of S^3, but possibly some element of $\mathcal{H}_g$ must be represented by a special deformation where the handlebody passes through an "exotic" embedding. For example, can the handlebody $\varphi_t(N)$ be at all times a special neighborhood of a graph? If the answer is affirmative, we may still ask whether $\varphi_t(N)$ for some t must pass through an "apparently knotted" embedding. In Figure 3, we give an example of a special neighborhood that looks very knotted but determines a

Heegaard splitting of S^3! In Chapter 5, Corollary 3 we shall show exactly how "nice" the embedding $\varphi_t(N)$ can be.

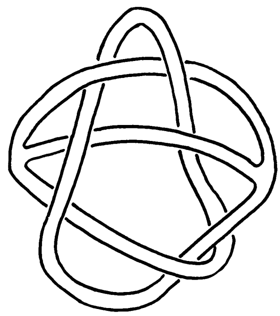

FIGURE 3. An unknotted handlebody in S^3.

The group operation in $\mathcal{H}_g$ corresponds to the composition of special deformations. Let D: $I \to H(S^3)$ and D': $I \to H(S^3)$ with $D(t) = \varphi_t$ and $D'(t) = \varphi'_t$ be special deformations; then the composition $D' \circ D$: $I \to H(S^3)$ with $D' \circ D(t) = \psi_t$ is defined by $\psi_t = \varphi_{2t}$ if $0 \leq t \leq \frac{1}{2}$ and $\psi_t = \varphi'_{2t-1} \circ \varphi_1$ if $\frac{1}{2} \leq t \leq 1$.

In summary, we have two different types of geometric representatives for an element $h \in \mathcal{H}_g$: (i) a homeomorphism $\varphi \in H(S^3, N)$, (ii) a special deformation D_φ: $I \to H(S^3)$ with $D_\varphi(t) = \varphi_t$ for $t \in I$. We are interested in those properties of homeomorphisms in $H(S^3, N)$ or of special deformations which depend only on the equivalence classes. For that reason we may assume that $D_\varphi(1) = \varphi$ and the images $\varphi(s_i)$ and $\varphi(\bar{s}_i)$ of the standard discs have a minimal number of intersections with the standard discs s_j, t_j, s'_j, s''_j, $\bar{s}_j$ and $\bar{t}_j$ for $j = 1, \ldots, g$. However, if φ can be chosen so that $\varphi(s_i) = s_k$ for some k, we shall always do so. If r and r' are any two sets of curves or discs, we shall use the symbol (r, r') for the number of components of the intersection of r and r'. These intersection numbers allow us to argue by induction. We shall move back and forth freely between the two types of representatives, depending on which representative is most useful. If $\varphi \in H(S^3, N)$ and a special deformation D_φ both represent $h \in \mathcal{H}_g$, we may denote D_φ by D_h and the element h by $[\varphi]$ or $[D_\varphi]$.

CHAPTER 2. THE GENERATORS OF $\mathcal{H}_g$

We shall use the special deformations introduced in Chapter 1 to describe five elements of the group $\mathcal{H}_g$. It will be shown in Chapter 4 that these elements generate $\mathcal{H}_g$. We shall also show that four of these elements generate a subgroup $\mathcal{P}_g$ of $\mathcal{H}_g$, which plays an important role in our argument.

The element ω is represented by a special deformation D_ω which is induced by a 180° rotation about the y axis of the 3-ball B_1 that contains the first handle. The deformation is fixed outside of a small neighborhood of B_1. See Figure 4a.

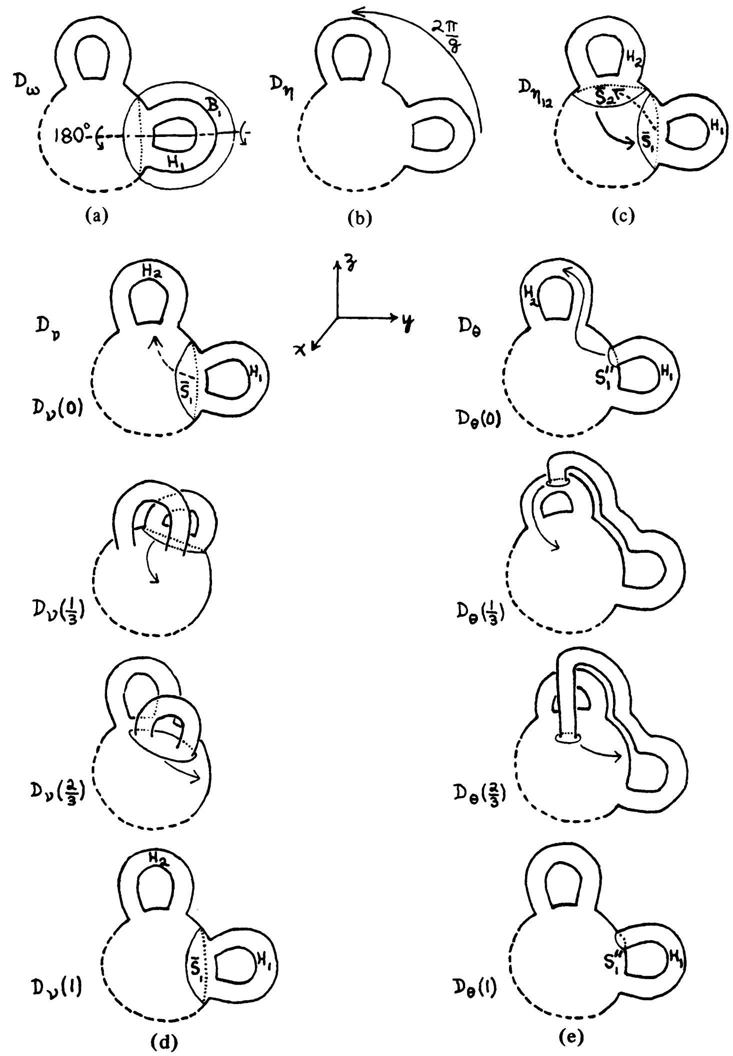

FIGURE 4. The generators of $\mathcal{H}_g$.

The element η is represented by a special deformation D_η which rotates S^3 about the x-axis by an angle of $2\pi/g$ taking the ith handle to the $(i + 1)$st handle. See Figure 4b.

The element η_{12} is represented by a special deformation D_η^{12} which exchanges the first handle and the second handle by letting the second handle pass in front of the first. See Figure 4c.

The element ν is represented by a special deformation D_ν which is obtained by sliding the whole first handle behind the first foot of the second handle, through the second handle and then back in front of the first foot of the second handle to its initial position. See Figure 4d.

The element ν is also represented by another special deformation D_ν' which leaves the first handle fixed, but is obtained by sliding the first foot of the second handle around the first handle so that it first passes in front, then goes behind the first handle. See Figure 5c. This simple example illustrates how seemingly different special deformations (such as D_ν and D_ν') are equivalent!

Finally, the element θ is represented by a special deformation D_θ which is obtained by sliding only the second foot of the first handle: slide it to the first foot of the second handle, up over the top of the second handle, and then back from the second foot of the second handle to its initial position by passing in front of the first foot of the second handle. See Figure 4e.

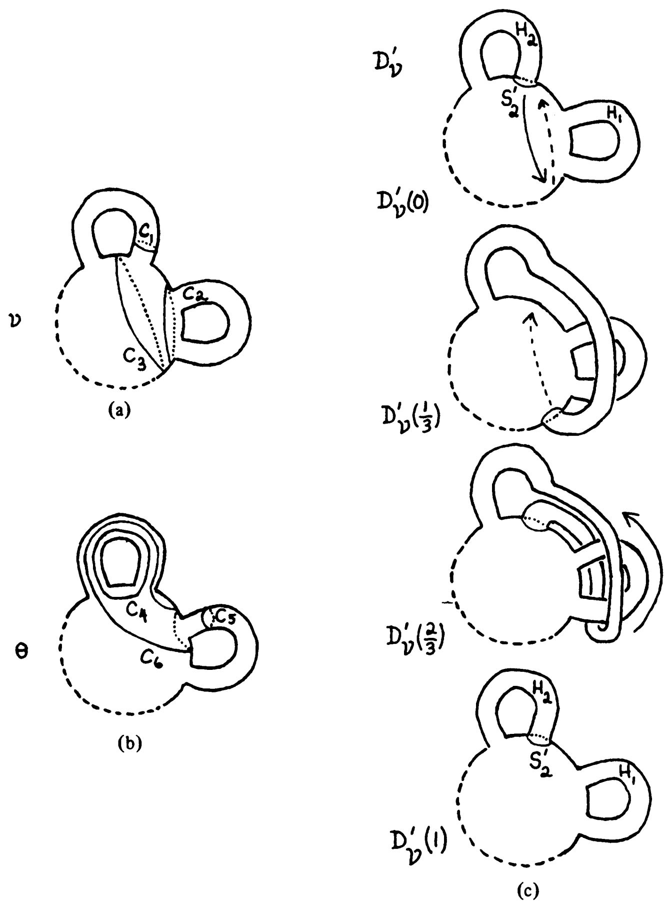

FIGURE 5. Other representations of ν and θ.

These five elements of the group $\mathcal{H}_g$ could also be defined by the homeomorphisms of the surface T that are determined by the special deformations. In particular, the element ν is represented by a product of three Dehn twists about the curves c_1, c_2, and c_3 in Figure 5a, i.e. $\nu = [\mathfrak{T}_{c_1}^{-1} \circ \mathfrak{T}_{c_2}^{-1} \circ \mathfrak{T}_{c_3}]$; the element θ is represented by a product of three Dehn twists about the curves c_4, c_5, and c_6 in Figure 5b, i.e. $\theta = [\mathfrak{T}_{c_4}^{-1} \circ \mathfrak{T}_{c_5}^{-1} \circ \mathfrak{T}_{c_6}]$.

Let $\mathcal{P}_g$ be the subgroup of $\mathcal{H}_g$ whose elements are represented by homeomorphisms which preserve the meridian discs of the handlebody N. Let $S = \{s_1, \ldots, s_g\}$ be the set of meridian discs of N. Then we have

$$\mathcal{P}_g = \langle [\varphi] \in \mathcal{H}_g \mid \varphi \in H(S^3, N, S) \rangle.$$

LEMMA 1. *$\mathcal{P}_g$ is generated by ω, η, η_{12}, and ν.*

PROOF. If $\varphi \in H(S^3, N, S)$ then φ leaves the set S invariant, therefore we may assume φ leaves the set of handles $\{H_1, \ldots, H_g\}$ invariant. Then φ is determined by its restriction $\bar{\varphi}$ to a homeomorphism of the boundary $\partial\overline{N}$ of the node of N. The map $\bar{\varphi}$ has the following properties: (i) $\bar{\varphi}$ preserves the set S' of feet of the handles $S' = \{s_1', s_1'', \ldots, s_g', s_g''\}$, (ii) $\bar{\varphi}$ extends to an element of $H(S^3, N)$. If two homeomorphisms, which restrict to homeomorphisms satisfying properties (i) and (ii), represent the same element in $\mathcal{P}_g$, $g > 1$, we may assume that during the isotopy properties (i) and (ii) are

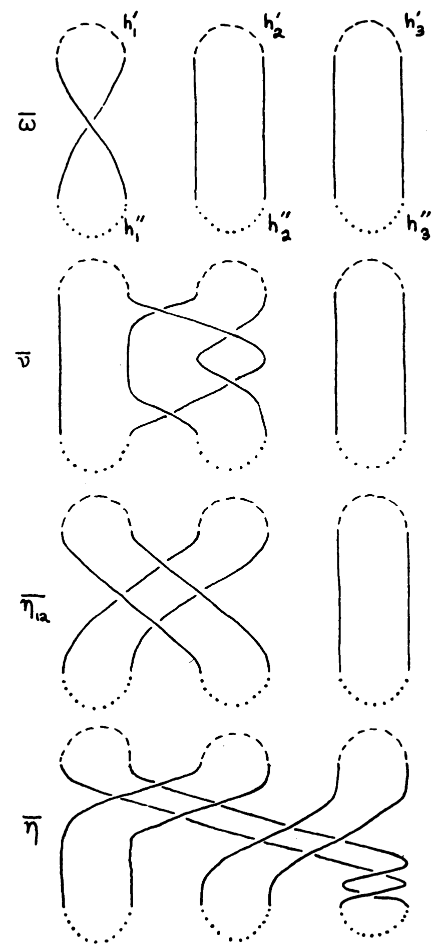

FIGURE 6. The trivial plat generators.

preserved (see [**Ha**]). Thus, $\mathcal{P}_g$ is isomorphic to the subgroup of $\pi_0(H(\partial\overline{N}, S'))$ consisting of elements whose representatives satisfy property (ii).

The group $\pi_0(H(\partial\overline{N}, S'))$ is isomorphic to the $2g$-string braid group of the 2-sphere [**B-1**, p.34] so we shall reinterpret property (ii) as a statement about braids. Regard D^2 as a subset of S^2. We may represent elements of the $2g$-string braid group of S^2 by geometric braids in $D^2 \times I$ visualized as a slice in 3-space. If we join the top ends of the $2i$th and the $(2i-1)$th strings by an arc h_i', as in Figure 6, we obtain g arcs which correspond to the handles. Then an element in $H(\partial\overline{N}, S')$ extends to an element in $H(S^3, N)$ if the corresponding braid has the property that the union of the braid and the arcs h_i' can be isotoped in S^3 to the union of the trivial braid and the arcs h_i', while keeping the base of the braid fixed. Let us also join the bottom ends of the $2i$th string and the $(2i-1)$st string by an arc h_i''. In this way we associate to each braid a *plat*. Property (ii) says simply that the plat thus obtained must be trivial.

Therefore, the group $\mathcal{P}_g$ is isomorphic to the subgroup of braids which have the property that the plat associated with the braid (as above) is isotopic in S^3 to the trivial plat. This subgroup has been studied by Hilden [**Hi**] who showed the braids pictured in Figure 6 generate this group. We denote these braids by the symbols $\bar{\omega}$, $\bar{\nu}$, $\bar{\eta}_{12}$, $\bar{\eta}$ and the corresponding elements of $\mathcal{P}_g$ are precisely ω, ν, η_{12}, and η. Q.E.D.

Chapter 3. Handle Slidings

The special deformations pictured in Figures 4a, 4d, 4e and 7 are examples of slidings of the first handle. We now describe the general setting for handle slidings. Denote the boundary of the standard handlebody without the ith handle by T^i. A special deformation is a sliding of the ith handle or an *i-sliding* if it is an extension of a deformation of the surface T^i. Therefore, if $D_\varphi\colon I \to H(S^3)$ is an i-sliding and $D_\varphi(t) = \varphi_t$, then $\varphi_t(T^i) = T^i$ for all $t \in I$. Since we also have $\varphi(N) = N$ (where $\varphi = \varphi_1$) the two feet of the ith handle must come back to their initial position as a set, i.e. $\varphi(s_i' \cup s_i'') = s_i' \cup s_i''$. Thus, an i-sliding is determined by a deformation of T^i that isotopes $s_i' \cup s_i''$ back to its initial position (as a set). But an arbitrary deformation of T^i that isotopes $s_i' \cup s_i''$ back to its initial position cannot be extended to a special deformation. The deformation necessarily preserves the standard handlebody without the ith handle, but the ith handle cannot always be isotoped back to its initial position in $S^3 - T^i$. We give an illustration of such a situation in Figure 8.

If we compose two i-slidings we obtain once again an i-sliding, so the elements represented by i-slidings form a subgroup of $\mathcal{H}_g$ that we shall denote $\mathcal{H}_g^i$. The aim of the chapter is to find generators for $\mathcal{H}_g^i$. For simplicity we restrict ourselves to the study of $\mathcal{H}_g^1$, because $\mathcal{H}_g^i = \eta^{1-i}\mathcal{H}_g^1\eta^{i-1}$. This study will be divided in two parts. In the first part we easily find generators for a subgroup of $\mathcal{H}_g^1$, but the second part is difficult.

1. *Slidings on a pedestal.* The 3-ball B_1 that contains the first handle

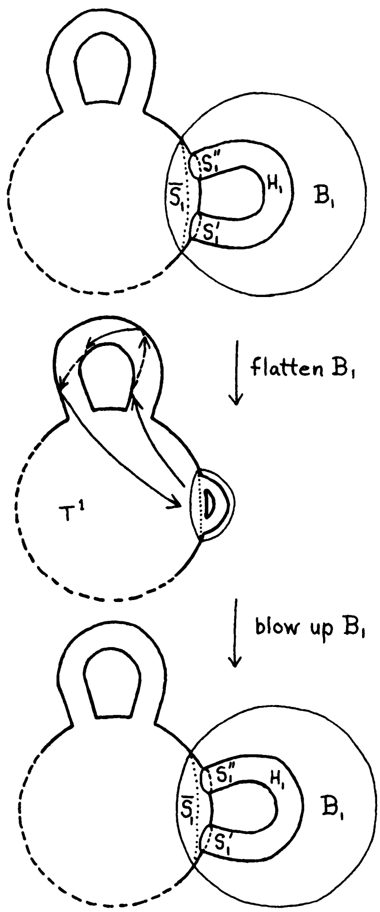

FIGURE 7. A 1-sliding on a pedestal.

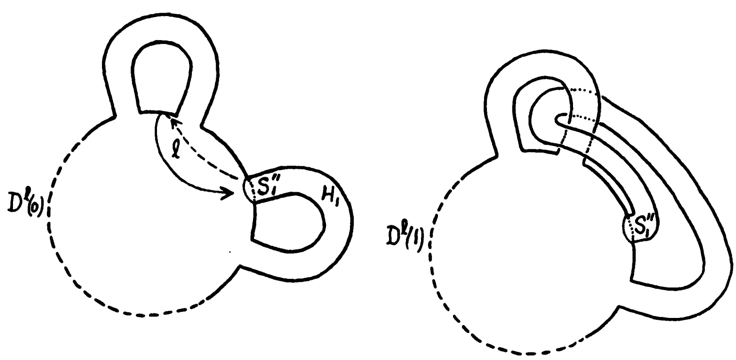

FIGURE 8. A sliding of a foot not extending to a special deformation.

intersects the surface T^1 in a disc $\bar{s}_1$, the pedestal of the first handle. Any deformation of T^1 that begins with the identity map and ends with a map that preserves $\bar{s}_1$ and $s_1' \cup s_1''$ can be extended to a special deformation as follows: first flatten the 3-ball B_1 so that it is in a small neighborhood of $\bar{s}_1$, then extend the deformation of T^1 to a deformation including the flattened 3-ball; finally, when $\bar{s}_1$ and $s_1' \cup s_1''$ are back to their initial positions, blow up B_1 to get a special deformation. These steps are illustrated in Figure 7. Let $\overline{\mathcal{H}}_g^1$ be the subgroup of $\mathcal{H}_g^1$ whose elements can be represented by these 1-slidings on a pedestal.

Let us denote by η_{ij} the element of $\mathcal{H}_g$ determined by a homeomorphism in $H(S^3, N)$ that exchanges the ith and jth handles so that the images of the loop a_j and b_j in Figure 2 are the loops a_i and b_i.

LEMMA 2. *The group $\overline{\mathcal{H}}_g^1$ is generated by the elements*

$$\{\omega, \nu, \omega^{-1}\theta\omega\theta, \eta_{2j}^{-1}\nu\eta_{2j}, \eta_{2j}^{-1}\omega^{-1}\theta\omega\theta\eta_{2j}; j = 3, \dots, g\}.$$

PROOF. The elements of $\overline{\mathcal{H}}_g^1$ are in one-to-one correspondence with the isotopy classes of deformations of the disc $\bar{s}_1$ (the pedestal) on the surface T^1 that bring $\bar{s}_1$ and $s_1' \cup s_1''$ back to their initial positions. Therefore, the group $\overline{\mathcal{H}}_g^1$ is isomorphic to the free product of the 1-string braid group of T^1 and the infinite cyclic group represented by rotations of $\bar{s}_1$ that leave $s_1' \cup s_1''$ invariant. The infinite cyclic group is generated by the 180° rotation of $\bar{s}_1$ that extends to a special deformation representing the element ω and the 1-string braid group is naturally isomorphic to the fundamental group of the surface T^1 **[B-1]**. Therefore, the other generators of $\overline{\mathcal{H}}_g^1$ are represented by 1-slidings of the pedestal $\bar{s}_1$ that describe the loops a_j and b_j, $j \neq 1$, which represent generators of the fundamental group of T^i. The elements $\omega^{-1}\theta\omega\theta$, ν, $\eta_{2j}^{-1}\omega^{-1}\theta\omega\theta\eta_{2j}$, and $\eta_{2j}^{-1}\nu\eta_{2j}$ represent the 1-slidings on the pedestal $\bar{s}_1$ about the loops a_2, b_2, a_j, and b_j, respectively, where $j = 3, \dots, g$. Q.E.D.

An arbitrary element in $\mathcal{H}_g^1$ shall be determined by a deformation of $s_1' \cup s_1''$ on T^i where $s_1' \cup s_1''$ (but not necessarily $\bar{s}_1$) is brought back to its initial position.

LEMMA 3. *The group $\mathcal{H}_g^1$ is generated by its subgroup $\overline{\mathcal{H}}_g^1$ together with the elements*

$$\{\theta, \eta_{2j}^{-1}\theta\eta_{2j}, \eta^{-1}\nu\eta, \eta_{ig}\eta^{-1}\nu\eta\eta_{ig}^{-1}; i = 2, \dots, g \text{ and } j = 3, \dots, g\}.$$

PROOF. Let l be a loop on $T^1 - \{s_1' \cup s_1''\}$ based at s''_1 and let D^l: $I \to H(S^3)$ be the extension to S^3 of an isotopy of the second foot s_1'' along the loop l. We first show that Lemma 3 can be reduced to studying maps of the type of D^l in the case where l bounds a disc in $\mathrm{Cl}(S^3 - N)$.

In the general case, the deformation D^l cannot be chosen to be a special deformation (for example, see Figure 8). However, if the loop l bounds a disc $\bar{l}$ in $\mathrm{Cl}(S^3 - N)$, then D^l can be chosen to be a special deformation because the handle H_1 can be deformed back to its initial position along the disc l while the feet are kept fixed. For example, such a disc exists in Figure 5c.

For any 1-sliding D_φ we construct a special deformation D_φ^l. Let t_1 be the disc in $\mathrm{Cl}(S^3 - N)$ that is in Figure 1, and let $\varphi \in H(S^3, N)$ be determined by the 1-sliding D_φ. Because φ leaves the first handle invariant, the boundary of the disc $\varphi(t_1)$ goes once over the first handle; therefore, let the loop l be the union of the arcs $\varphi(t_1) \cap T^1$ and $t_1 \cap T^1$. Then l bounds a disc (possibly singular) in $\mathrm{Cl}(S^3 - N)$ that we may assume is based at s_1''. If D_φ^l is the 1-sliding of the second foot determined by l, then the composition $D_\varphi^l \circ D_\varphi$ determines an element in $\mathcal{K}_g^1$. Since the pedestal $\bar{s}_1$ is a regular neighborhood in T^1 of $s_1' \cup s_1'' \cup (t_1 \cap T^1)$, D_φ^l isotopes $\varphi(\bar{s}_1)$ back to $\bar{s}_1$ in T^1.

Therefore, we now see that Lemma 3 reduces to showing that each deformation D^l, where l bounds a disc $\bar{l}$ in $\mathrm{Cl}(S^3 - N)$ and is based at s_1'', represents a product of the indicated generators.

Case 1. We first show that we may assume the disc $\bar{l}$ does not pass under the first handle.

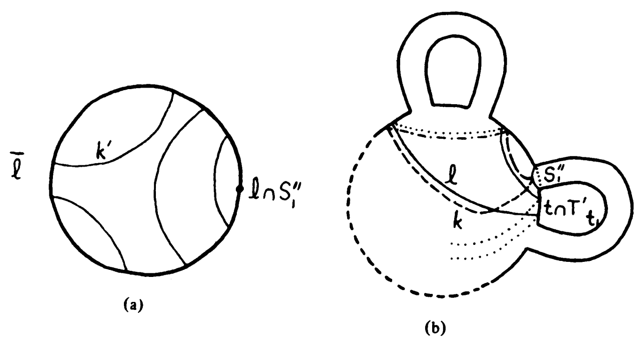

FIGURE 9. Simplifying a sliding of a foot (Case 1).

Suppose $(\bar{l}, t_1) > 0$. Let k' be an outermost arc of $\bar{l} \cap t_1$ on the disc $\bar{l}$; also choose k' so that the boundary of the disc $\bar{l}$ facing k' does not contain the base point $l \cap s_1''$, as illustrated in Figures 9a and 9b. We may form the loop k, starting at the base point, following the arc $t_1 \cap T^1$ to the first intersection with the arc k', then following the arc on the loop l that faces k' to the other intersection of k' and $t_1 \cap T^1$ and, finally, following $t_1 \cap T^1$ back to the base point. The loop k bounds a disc $\bar{k}$ in $\mathrm{Cl}(S^3 - N)$ such that $(\bar{k}, t_1) = 0$ and the image of $\bar{l}$ under the composition of the 1-sliding D^l by D^k has fewer intersections with t_1 than did $\bar{l}$. Thus, any 1-sliding D^l is a product of 1-slidings of the second foot around loops that bound discs disjoint from t_1.

Case 2. From now on we shall assume that $(\bar{l}, t_1) = 0$. The easiest case to handle shall be the case when the disc $\bar{l}$ does not pass over or through any of the handles. We make this precise.

The boundary of the 3-ball B_i containing the ith handle intersects $\mathrm{Cl}(S^3 - N)$ in a disc $\bar{t}_i$, as illustrated in Figure 1. Let $\overline{T}_1 = \{\bar{t}_2, \ldots, \bar{t}_g\}$. Suppose $(\bar{l}, \overline{T}_1) = 0$. Then l is a loop on the boundary of the node $\overline{N}$ disjoint from the pedestals $\bar{s}_2, \ldots, \bar{s}_g$ and $t_1 \cup s_1'$. The isotopy classes of such loops represent elements of the fundamental group of a 2-sphere with g holes. The 1-slidings of the second foot about loops around $t_1 \cup s_1'$, $\bar{s}_g, \bar{s}_2, \ldots, \bar{s}_{g-1}$ determine the following elements of $\mathcal{K}_g^1$ $\{\omega^2, \eta^{-1}\nu\eta$ and $\eta_{ig}\eta^{-1}\nu\eta\eta_{ig}^{-1}$; $i = 2, \ldots, g-1\}$.

Thus, any 1-sliding D^l with $(\bar{l}, t_1) = (\bar{l}, \overline{T}_1) = 0$ represents a product of the indicated generators of $\mathcal{H}_g^1$.

Case 3. We now treat the more difficult situation where the second foot s_1'' slides over or through other handles, i.e. $(\bar{l}, \overline{T}_1) > 0$. The set $\bar{l} \cap \overline{T}_1$ will consist of pairwise disjoint arcs properly embedded in the disc $\bar{l}$. The arcs subdivide the disc into a union of discs and one of these discs $\bar{d}_l$ has the base point of l on its boundary, as illustrated in Figure 10a.

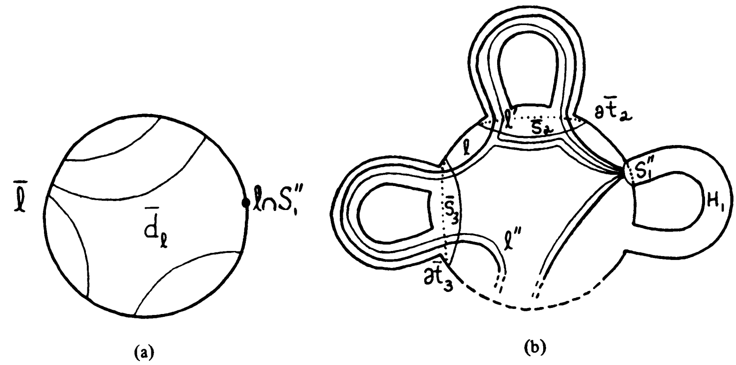

FIGURE 10. Simplifying a sliding of a foot (Case 3).

Suppose $(l, \bar{d}_l) > 1$. As in Figure 10b, we may decompose the 1-sliding D^l into two 1-slidings $D^{l'}$ and $D^{l''}$ that are extensions of the isotopies of the second foot of H_1 about the loops l' and l'' that we now describe: the loop l' starts at the base point, follows l until l intersects the disc $\bar{d}_l$ again, then l' follows the boundary of $\overline{T}_1$ back to the first intersection of l and $\overline{T}_1$, finally, l' follows l back to the base point. The loop l'' starts at the base point, follows the loop l' back to where it first left the loop l and then continues along l back to the base point. By construction l' and l'' bound discs so that the subdiscs $\bar{d}_{l'}$ and $\bar{d}_{l''}$ satisfy $(l_1' \bar{d}_{l'}) = 1$ and $(l'', \bar{d}_{l''}) < (l, \bar{d}_l)$; moreover, $D^{l'}$ and $D^{l''}$ are 1-slidings such that $[D^l] = [D^{l'} \circ D^{l''}]$.

Thus, it suffices to show that 1-slidings of the type of D^l with $(\bar{l}, t_1) = 0$, $(\bar{l}, \overline{T}_1) > 0$ and $(l, \bar{d}_l) = 1$ represent products of the elements of $\mathcal{H}_g^1$ listed in Lemma 3.

Case 4. Suppose $(\bar{l}, t_1) = 0$, $(\bar{l}, \overline{T}_1) > 0$, and $(l, \bar{d}_l) = 1$. By composing D^l with 1-slidings of the second foot which leave the 3-balls $B_2, \ldots, B_g$ fixed and satisfy Case 2, we may assume that the two arcs of the loop l that join the base point of l to the first intersection with $\overline{T}_1$ are parallel as in Figure 11. Then the loop l can only pass through the ball B_i in four possible ways. First, both arcs of the loop l can go over the ith handle, as illustrated in Figure 11a. Second, both arcs can go under the ith handle, as illustrated in Figure 11b. In these first two cases, a 1-sliding of the pedestal $\bar{s}_1$ following the arcs over or under the handle and back reduces the number of intersections of l and $\overline{T}_1$.

Third, one of the arcs of the loop l may go over the ith handle and the other through the ith handle, as illustrated in Figure 11c. If we compose D^l

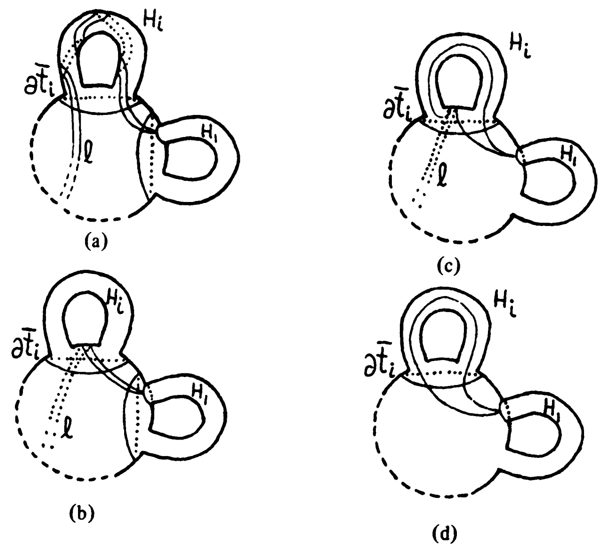

FIGURE 11. Simplifying a sliding of a foot (Case 4).

with a 1-sliding D^{a_i} of the second foot about the loop a_i, we return to the second situation without increasing $(l, \bar{T}_1)$.

Finally, the loop l may already be the loop a_i, as in Figure 11d.

In Case 4, the only new generators that we introduce are represented by D^{a_i} for $i = 2, \ldots, g$. The 1-sliding D^{a_2} represents θ and the 1-sliding D^{a_i} represents $\eta_{2i}^{-1}\theta\eta_{2i}$, $i = 3, \ldots, g$. Using these generators, we may change D^l_φ to a 1-sliding which satisfies Case 2. This proves Lemma 3.

In the next chapter, handle slidings will enable us to simplify the elements of the group $\mathcal{H}_g$.

CHAPTER 4. $\mathcal{H}_g$ IS FINITELY GENERATED

Let $\mathcal{H}_g^*$ be the group generated by slidings of all the g handles of the handlebody N. We shall prove that the group $\mathcal{H}_g$ is generated by the groups $\mathcal{P}_g$ and $\mathcal{H}_g^*$.

THEOREM. *If $h \in \mathcal{H}_g$, then $h = h_2 \circ h_1$ where $h_1 \in \mathcal{P}_g$ and $h_2 \in \mathcal{H}_g^*$.*

COROLLARY 1. *$\mathcal{H}_g$ is generated by the elements $\{\omega, \nu, \eta, \eta_{12}, \theta\}$.*

PROOF OF COROLLARY 1. If the Theorem is true then $\mathcal{H}_g$ is generated by the groups $\mathcal{P}_g, \mathcal{H}_g^1, \ldots, \mathcal{H}_g^g$ and Lemmas 1, 2, and 3 give us these generators for $\mathcal{H}_g$.

PROOF OF THE THEOREM. We first introduce graphs. Let $h \in \mathcal{H}_g$ and $\varphi \in H(S^3, N)$ represent the element h. We associate to h the *graph* $G^h = \varphi^{-1}(G)$, where G is the standard graph. In Figure 12 we give an example of a graph G^h, where h is represented by a 1-sliding (altering the edge e_1). The basic idea behind our proof of the theorem will be to show that (i) each graph G^h is the image of the standard graph G under a product of a finite number of handle slidings, and (ii) if $G^h = G$ then h is in the subgroup $\mathcal{P}_g$.

The graph G^h has g edges $e_1^h, \ldots, e_g^h$ which are the images under φ^{-1} of the edges $e_1, \ldots, e_g$ of the standard graph G. Recall that S is the set $\{s_1, \ldots, s_g\}$ of meridian discs of N. We assume that the homeomorphism φ was chosen in its isotopy class so that (G^h, S) is minimal and so that φ leaves the vertex of G invariant. We now relate the graph G^h to the image of the set S under the homeomorphism φ^{-1}.

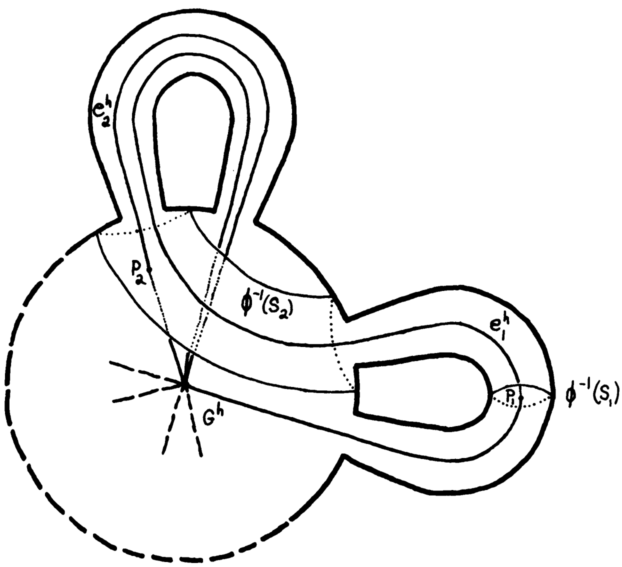

FIGURE 12. The associated graph and $\varphi^{-1}(S)$.

LEMMA 4. *Let $\varphi \in H(S^3, N)$ and $h = [\varphi]$. Then the graph G^h and the set $\varphi^{-1}(S)$ determine one another up to isotopy in the handlebody N.*

PROOF. The edges of G^h intersect the set $\varphi^{-1}(S)$ as follows: $(e_i^h, \varphi^{-1}(s_i)) = 1$ and $(e_i^h, \varphi^{-1}(s_j)) = 0$ for $j \neq i$ where $i, j \in \{1, \ldots, g\}$, because by the construction of the graph $(e_i^h, \varphi^{-1}(s_j)) = (\varphi^{-1}(e_i), \varphi^{-1}(s_j)) = (e_i, s_j)$.

Let us first suppose that $\varphi^{-1}(S)$ is given. We may then construct G^h: first, pick a point p_i on the disc $\varphi^{-1}(s_i)$ through which e_i^h shall pass; if N is cut open along $\varphi^{-1}(S)$ we obtain a 3-ball and the point p_i determines two points p_i' and p_i'' on the boundary of the 3-ball. Now join the $2g$ points just obtained to a vertex by arcs so that the 3-ball is a regular neighborhood of the $2g$ arcs, then the arcs are unique up to isotopy. If we re-identify the copies of $\varphi^{-1}(S)$ we obtain G^h. See Figure 12 for an example of this construction.

We now suppose that G^h is given. Let the graph G_i^h be the graph G^h minus the edge e_i^h. Then the handlebody N cut open along $\varphi^{-1}(s_i)$ is a regular neighborhood of the graph G_i^h because the standard handlebody cut open along s_i is a regular neighborhood of the graph G minus the edges e_i. Therefore, there is a unique disc in N up to isotopy that is disjoint from G_i^h

and it must be the disc $\varphi^{-1}(s_i)$. This construction of $\varphi^{-1}(S)$ concludes the proof of Lemma 4.

We now use these graphs to prove the theorem by an induction on the intersection numbers (G^h, S).

Since the standard handlebody is a regular neighborhood of the graph G, it must also be a regular neighborhood of the graph G^h because $G^h = \varphi^{-1}(G)$ and $N = \varphi^{-1}(N)$. Therefore, the graph G^h must have an edge going through each handle, i.e. $(G^h, s_i) \geqslant 1$, and thus, for any $h \in \mathcal{H}_g$, $(G^h, S) \geqslant g$.

Step 1. Suppose $(G^h, S) = g$. Then, by Lemma 4, exactly one edge of G^h must go through each handle, otherwise the standard handlebody N would not be a regular neighborhood of G^h (there would be a handle through which no edge passed). Therefore, we may assume that $\varphi^{-1}(S) = S$ and thus, h is an element in the subgroup $\mathcal{P}_g$.

Step 2. Suppose $(G^h, S) = n$ for some n greater than g. By induction, we assume that $h = h_2 \circ h_1$ where $h_1 \in \mathcal{P}_g$ and $h_2 \in \mathcal{H}_g^*$ (i.e. h_2 is a product of handle slidings) for all $h \in \mathcal{H}_g$ such that $(G^h, S) < n$.

If we can find a k-sliding D_ψ: $I \to H(S^3)$ such that $h' = [D_\psi]$ and $(G^{h' \circ h}, S) < (G^h, S)$, then $h' \circ h$ satisfies the induction hypothesis, hence $h' \circ h = h_2' \circ h_1'$ where $h_2' \in \mathcal{H}_g^*$ and $h_1' \in \mathcal{P}_g$. Therefore, $h = h'^{-1} \circ h' \circ h = h'^{-1} \circ h_2' \circ h_1'$ satisfies our Theorem, thus proved by induction.

To find the k-sliding D_ψ we shall first construct a new graph G' obtained from G^h by changing just one edge such that G' is also a retract of the handlebody N and such that $(G', S) < (G^h, S)$. Later we shall find a k-sliding that takes G^h to G'.

Choosing an edge e_k^h of G^h which goes through more handles of N than the other edges, i.e. $(e_k^h, S) \geqslant (e_l^h, S)$ for $l \neq k$.

Claim. We can find an edge e' with the following properties: (i) $e' \subseteq N$ and the boundary of e' is the vertex of G^h, (ii) $(e', \varphi^{-1}(s_k)) = 1$, (iii) the edge e' bounds a disc in $S^3 - G_k^h$ (recall $G_k^h = G^h - e_k^h$), and, finally, (iv) $(e', S) < (e_k^h, S)$, so that $G' = G_k^h \cup e'$ is a retract of the handlebody N.

The construction of the edge e' is the most difficult part of the proof. If we obtain such an edge e' then we have $(G', S) < (G^h, S)$ because $(G', S) = (G_k^h, S) + (e', S) < (G_k^h, S) + (e_k^h, S) = (G^h, S)$. It shall then remain to prove that there is a k-sliding which transforms G^h to G', but this shall not be hard.

First, we will construct an arc α that shall be used in the construction of the edge e'. Let p be the intersection of the edge e_k^h and the disc $\varphi^{-1}(s_k)$ and let p' and p'' be the two copies of p in the standard handlebody N opened along the disc $\varphi^{-1}(s_k)$. Let R be a retraction of N opened along the disc s_k onto $G - e_k$. There are many such retractions and during the proof we shall modify the retraction R. (Note that R does not induce a retraction of N onto the graph G.) Now $\varphi^{-1}R\varphi$ is a retraction of $N - \varphi^{-1}(s_k)$ onto G_k^h. During the retraction $\varphi^{-1}R\varphi$ the points p' and p'' each describe an arc in N ending on an edge of G_k^h or on the vertex of G_k^h. Let the arc α be the union of the two arcs when p' and p'' are re-identified.

We first show that (α, S) can be assumed to be 0 or 1. Intuitively, this means that although the arc α may be quite messy, we may push α off all the handles, except possibly off one handle if $\varphi^{-1}(s_k) = s_i$ for some i. We will now make this precise. There are three cases.

Case 1. We first assume that the disc $\varphi^{-1}(s_k)$ is disjoint from the handles, but that $\varphi^{-1}(s_k)$ is not isotopic to a meridian disc s_i. Such is the disc $\varphi^{-1}(s_2)$ in the example given in Figure 12. Let p be the intersection of e_k^h and $\varphi^{-1}(s_k)$. The disc $\varphi^{-1}(s_k)$ cuts open the node $\bar{N}$ into two components, one of which contains the vertex of G_k^h and the point p', and the other component of the node must intersect at least one edge, say, e_m^h, of G_k^h and contain the point p''. We would like to arrange matters so that p' is retracted onto the vertex of G_k^h, p'' is retracted onto the edge e_m^h and so that α is entirely contained in the node $\bar{N}$. In general, $\varphi^{-1}R\varphi$ shall not be such a retraction, but it is clear that we can construct such a retraction of $N - \varphi^{-1}(s_k)$ onto G_k^h and, therefore, we could have chosen R so that the retraction $\varphi^{-1}R\varphi$ determines an arc α entirely contained in the node $\bar{N}$ (i.e. $(\alpha, S) = 0$).

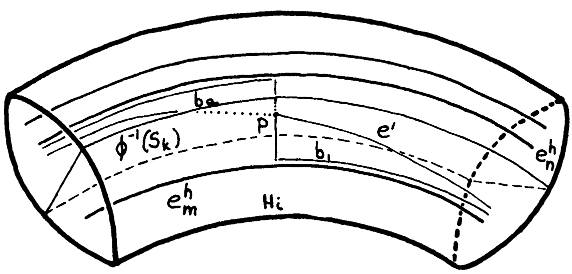

FIGURE 13. Construction of the edge e' in the handle.

Case 2. Suppose the disc $\varphi^{-1}(s_k)$ intersects a handle, i.e. $(\varphi^{-1}(s_k), S) > 0$. This case is illustrated in Figure 13. We may assume that the disc $\varphi^{-1}(s_k)$ and the edges of G_k^h are parallel to the cores of the handles, and that the point p of $\varphi^{-1}(s_k)$ is in one of the handles. then, if the handle is cut open along $\varphi^{-1}(s_k)$, the components containing the points p' and p'' (resp.) must also contain part of an edge of G_k^h, say, the edge e_m^h and e_n^h (resp.)). It is possible that $m = n$. We may construct a retraction from $N - \varphi^{-1}(s_k)$ to G_k^h so that α is perpendicular to the core of the handle and joins e_m^h to e_n^h and we may assume that $\varphi^{-1}R\varphi$ is such a retraction. Again, we have $(\alpha, S) = 0$.

Case 3. We are now left with the case when $\varphi^{-1}(s_k)$ is a meridian disc of some handle. We may then choose the retraction $\varphi^{-1}R\varphi$ so that both points p' and p'' are retracted onto the vertex of G_k^h without going through any other handles. The points p' and p'' are in a situation similar to the point p' in Case 1. In this last case we obtain an arc α such that $(\alpha, S) = 1$.

We now construct the edge e' from α by joining the endpoints of the arc α to the vertex following subarcs of the edges e_m^h and e_n^h. In Case 1, one endpoint of α is already on the vertex and in Case 3 both endpoints of α are on the vertex. If an endpoint of α divides the edge e_m^h (resp. e_n^h) into two subarcs, let b_1 (resp. b_2) be the subarc that passes through the handles the least number of times, i.e. $(b_1, S) \leqslant \frac{1}{2}(e_m^h, S)$ and $(b_2, S) \leqslant \frac{1}{2}(e_n^h, S)$. Let the edge e' be a push-off of $b_1 \cup \alpha \cup b_2$ in $N - G_h^k$.

By construction the edge e' satisfies properties (i) and (ii) of the claim. We will now show that we may assume e' satisfies properties (iii) and (iv).

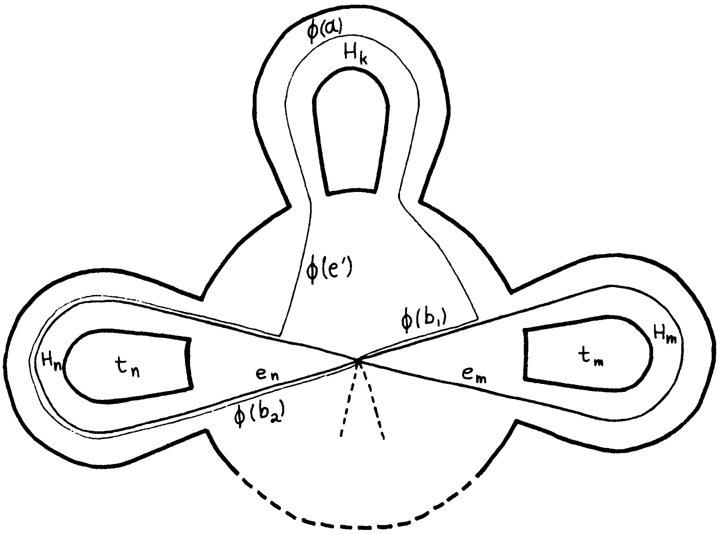

FIGURE 14. The image of the graph G' under φ.

We will first show that e' bounds a disc in $S^3 - G_k^h$. Consider the image of G' under the homeomorphism $\varphi \in H(S^3, N)$, $\varphi(G') = \varphi(G_k^h) \cup \varphi(e') = (G - e_k) \cup \varphi(e')$. Since φ is a map of S^3, it is adequate to show that the edge $\varphi(e')$ bounds a disc in $S^3 - (G - e_k)$ (see Figure 14). The edge $\varphi(e')$ is isotopic to the union of the arcs $\varphi(b_1)$, $\varphi(\alpha)$, and $\varphi(b_2)$, and $\varphi(\alpha)$ joins the edges e_m and e_n.

We may assume that $\varphi(\alpha)$ does not intersect the standard discs with boundaries $e_1, \ldots, e_{k-1}, e_{k+1}, \ldots, e_g$. We shall denote these discs with the symbols $t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_g$, though this is a slight abuse of notation. Since α is defined by the retraction $\varphi^{-1}R\varphi$ of the points p' and p'' which are disjoint from the discs $\varphi^{-1}(t_i)$ for $i \neq k$, we may assume that during the retraction the images of p' and p'' do not intersect $\varphi^{-1}(t_i)$ for $i \neq k$. Therefore, $\varphi(\alpha)$ is disjoint from the discs $t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_g$.

Now $\varphi(\alpha)$ is not a knotted arc, because $(G - e_k) \cup \varphi(\alpha)$ must be a retract of N by the construction of α. Therefore, if we had chosen the subarcs b_1 and b_2 to be disjoint from the discs $\varphi^{-1}(t_m)$ and $\varphi^{-1}(t_n)$, respectively, which is always possible, then $\varphi(e')$ is disjoint from the discs t_i, $i \neq k$, and bounds a disc in $S^3 - (G - e_k)$, hence e' bounds a disc in $S^3 - G_k^h$.

It remains now to show that the edge e' satisfies property (iv), $(e', S) < (e_k^h, S)$. If we are in Case 3, $(e', S) = 1$ and $(e_k^h, S) > 1$ by induction. If we are in Case 1, $(e', S) \leq (b_1, S) + (\alpha, S) \leq \frac{1}{2}(e_m^h, S)$ and, therefore, $(e', S) < (e_k^h, S)$. Finally, in Case 2, $(e', S) \leq \frac{1}{2}(e_m^h, S) + \frac{1}{2}(e_n^h, S)$, so $(e', S) < (e_k^h, S)$ unless $(e_k^h, S) = (e_m^h, S) = (e_n^h, S)$. We have this last equality when the point p is in the ith handle where the arc α is transverse to the handle, and the boundary of α divides e_m^h (resp. e_n^h) into subarcs each of which contains exactly half the points of $e_m^h \cap S$ (resp. $e_n^h \cap S$). See Figure 15. If we choose the subarcs b_1 and b_2 so that they both intersect the meridian disc s_i as illustrated in Figure 15, then by an isotopy of e' we obtain $(e', S) < (e_k^h, S)$. By construction G' is a retract of N and, thus, we have proved the claim.

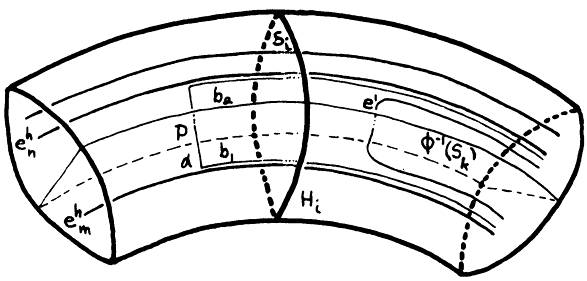

FIGURE 15. Isotopy of e' to reduce (e', S).

We now have two retracts of the handlebody N, the graphs G^h and G' which have G_k^h as a subgraph and where the edge e_k^h in G^h is replaced by the edge e' in G'. A retraction of the handlebody N onto the graph G' will determine a "sliding" of the edge e_k^h to the edge e'; since the edges e_k^h and e' have the point p in common and only intersect $\varphi^{-1}(s_k)$ once, a retraction of N onto G' will leave G_k^h invariant and deform e_k^h onto the graph G' so that it passes only once over the edge e'; the ends of the edge e_k^h will slide over the edges of G_k^h as e_k^h is deformed to e'. Now the image of this "sliding" of e_k^h under the homeomorphism φ will determine a "sliding" of the edge e_k on the graph $G - e_k$ to the edge $\varphi(e')$. Without moving the endpoints of $\varphi(e')$ we may now deform $\varphi(e')$ back to e_k in the 3-sphere because $\varphi(e')$ bounds a disc in $S^3 - (G - e_k)$. The thickening of the "sliding" of e_k and of the deformation of $\varphi(e')$ to special neighborhoods determines a k-sliding! Denote the k-sliding $D_\psi\colon I \to H(S^3)$ and let $h' = [D_\psi]$. Then $G' = G^{h' \circ h}$ and $(G^{h' \circ h}, S) < (G^h, S)$ where $h' \circ h = [D_\psi \circ D_\varphi]$. This proves the Theorem.

REMARK. The only elements of the subgroup $\mathcal{H}_g$ not in $\mathcal{H}_g^*$ are those elements that permute the handles. Thus, the group $\mathcal{H}_g$ is generated by the group of handle slidings $\mathcal{H}_g^*$ and the group of permutations of the handles $\langle \eta, \eta_{12} \rangle$.

CHAPTER 5. A SECOND SET OF GENERATORS FOR $\mathcal{H}_g$

In this chapter we define a second set of generators for $\mathcal{H}_g$. If we regard N as a neighborhood of the disc D_g with g holes, $D_g = N \cap \{y\text{-}z \text{ plane}\}$, then there is a natural subgroup $\mathcal{B}_g$ of $\mathcal{H}_g$, induced by isotopy classes of homeomorphisms of the g-punctured disc D_g which is isomorphic to the braid group on g-strings [**B-1**, p. 17].

Let $\sigma_1, \ldots, \sigma_{g-1}$ be the generators of $\mathfrak{B}_g$ where σ_i corresponds to the crossing of the ith and the $(i + 1)$st string and is represented by a special deformation, D_{σ_i}, which exchanges the ith hole t_i and the $(i + 1)$st hole t_{i+1} by having t_i pass over t_{i+1} as in Figure 16. Any element in $\mathfrak{B}_g$ is then represented by a special deformation where N is kept on the y-z plane.

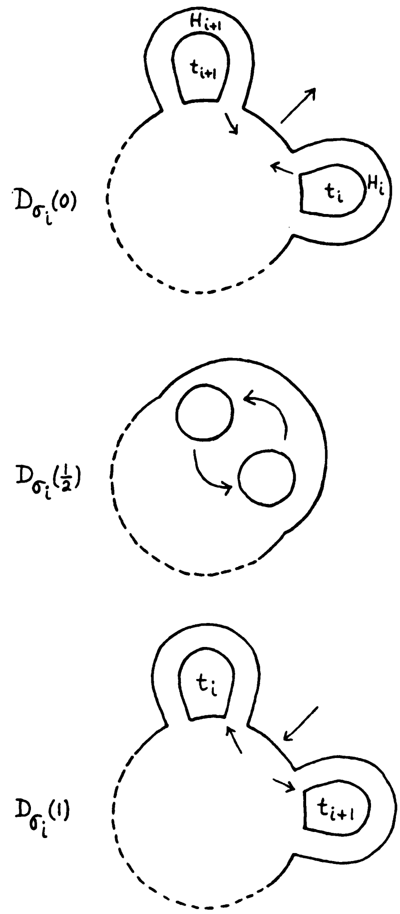

FIGURE 16. Special deformation representing σ_i.

The three other generators are described by special deformations to be called flips.

The generator ω described in Figure 4a is a flip of the first handle.

The generator ω' described in Figure 17a is a flip of the first and second handles. Let B'_{12} be a 3-ball containing the first and second handles so that the boundary of B'_{12} is divided by the surface T into two discs, one in N and one in $\mathrm{Cl}(S^3 - N)$. Then the generator ω' is represented by a special deformation $D_{\omega'}$ induced by a 180° rotation of B'_{12} about an axis passing through the centers of the two discs forming the boundary of B'_{12}.

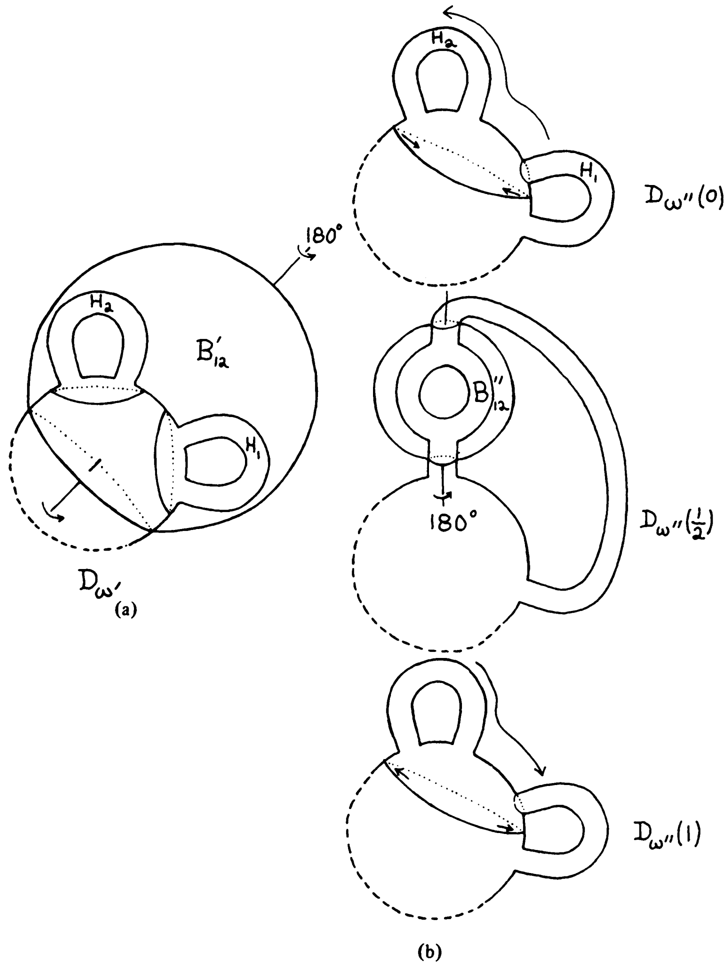

FIGURE 17. Special deformations representing ω' and ω''.

The generator ω'' described in Figure 17b is a flip which also involves the first and second handles, but to perform the flip we must deform the handlebody N into a different position. The special deformation $D_{\omega''}$ first slides the second foot of the first handle up to the top of the second handle and squeezes the pedestal of the second handle so that there is a 3-ball B''_{12} whose boundary intersects the handlebody in two discs and contains a component of genus 1; then B''_{12} is flipped by 180° about an axis passing through the centers of the two discs, after which the handles are brought back to their initial position.

COROLLARY 2. $\mathcal{H}_g = \langle \sigma_1, \ldots, \sigma_{g-1}, \omega, \omega', \omega'' \rangle$.

PROOF OF COROLLARY 2.

$$\eta \in \mathfrak{B}_g, \quad \eta_{12} = \omega^{-1}\omega'\omega^{-1}, \quad \nu = (\omega'\omega^{-1}\omega')(\omega'')(\omega'\omega^{-1}\omega')$$

and

$$\theta = \omega'\omega^{-1}\omega'\omega''$$

(check the compositions of special deformations).

In general, a *flip* of N is induced by a 180° rotation of a 3-ball whose boundary intersects N in one or two discs transverse to the y-z plane. The axis of the rotation must pass through the center of the disc or discs. This enables us to define very simple special deformations that determine all elements of $\mathcal{H}_g$.

COROLLARY 3. *If $h \in \mathcal{H}_g$, then h is represented by a special deformation D_h which is a composition of two types of deformations of the handlebody N: deformations keeping the handlebody on the y-z plane (so that it is always a special neighborhood of a graph embedded in the y-z plane), and flips.*

REMARK. As noted in the introduction, if φ is a homeomorphism of S^3 which fixes a Heegaard handlebody N then φ is isotopic to the identity map via an isotopy that moves N. The image of N during this isotopy then gives a continuous sequence of embeddings of N in S^3 that, at various stages, may look knotted, although they are, of course, unknotted (i.e. the complement of N in S^3 has a free fundamental group). Corollary 3 says that any map φ may be isotoped back to the identity map through a sequence of special isotopies, each of which is either a flip or a deformation which preserves the property that N is a special neighborhood of a spine in the y-z plane. Thus, we may always find an isotopy of φ to the identity map which avoids apparently knotted embeddings.

CHAPTER 6. $\mathcal{H}_g$ AND OUT$(\pi_1(N))$

We now give some brief remarks on the natural homomorphism from $\mathcal{H}_g$ to Out$(\pi_1(N))$. If $h \in \mathcal{H}_g$, then h is represented by a homeomorphism φ in $H(S^3, N)$ such that $[\varphi] = h$, and the restriction of φ to a homeomorphism of N induces an automorphism of $\pi_1(N)$. Therefore, there is a homomorphism, which we call μ, from $\mathcal{H}_g$ to Out$(\pi_1(N))$. Let $\mathcal{K}_g$ denote the kernel of μ.

We first show that μ is surjective. The group $\pi_1(N)$ is free and the edges $e_1, \ldots, e_g$ of the graph G represent a set of generators $a_1, \ldots, a_g$ for $\pi_1(N)$. Now, the subgroup $\mathcal{P}_g$ of $\mathcal{H}_g$ is represented by homeomorphisms that preserve G. Therefore, elements in the subgroup $\mathcal{P}_g$ induce automorphisms of $\pi_1(N)$ that are generated either by permutations of the generators or by replacing any generator a_i by its inverse, because the images of the standard loops are again the edges of the graph G.

For the generator θ of $\mathcal{H}_g$ the image of the standard loops representing the generators $a_1, \ldots, a_g$ is the graph G^h in Figure 12. The image $\mu(\theta)$ of θ under the homomorphism μ is an elementary Nielsen transformation of the free group $\pi_1(N)$, namely, $\mu(\theta)(a_i) = a_i$ if $i \neq 1$ and $\mu(\theta)(a_1) = a_1a_2$. The automorphisms induced by the group $\mathcal{P}_g$ and θ generate all automorphisms of $\pi_1(N)$, thus, μ is surjective.

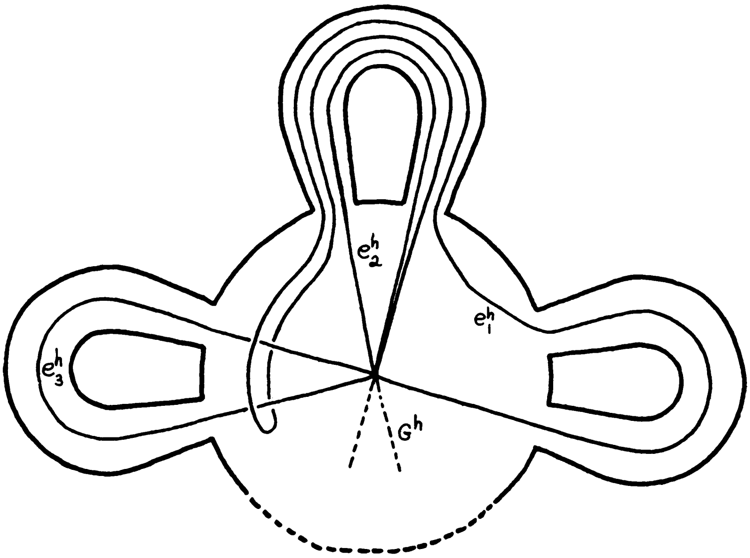

FIGURE 18. Associated graph of an element in $\mathcal{K}_g$.

An interesting problem is to find a finite set of generators for $\mathcal{K}_g$. In Figure 18 we give an example of a 1-sliding which is in the kernel $\mathcal{K}_g$. If $h \in \mathcal{K}_g$, then the graph G^h associated with h must be homotopic to the standard graph G in N, because G^h is the image of the standard loops $e_1, \ldots, e_g$ which represent the generators of $\pi_1(N)$. But, as illustrated in Figure 18, G^h may be homotopic, but not isotopic, to the standard graph in N. (Is there such an example for genus 2?) A better understanding of the graph associated to the elements of $\mathcal{K}_g$ should help to find generators for $\mathcal{K}_g$.

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